

GO-SPACES AND NOETHERIAN SPECTRA

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ABSTRACT. The Noetherian type of a space is the least κ for which the space has a κ^{op} -like base, *i.e.*, a base in which no element has κ -many supersets. We prove some results about Noetherian types of (generalized) ordered spaces and products thereof. For example: the density of a product of not-too-many compact linear orders never exceeds its Noetherian type, with equality possible only for singular Noetherian types; we prove a similar result for products of Lindelöf GO-spaces. A countable product of compact linear orders has an ω_1^{op} -like base if and only if it is metrizable, and every metrizable space has an ω^{op} -like base. An infinite cardinal κ is the Noetherian type of a compact LOTS if and only if $\kappa \neq \omega_1$ and κ is not weakly inaccessible. There is a Lindelöf LOTS with Noetherian type ω_1 and there consistently is a Lindelöf LOTS with weakly inaccessible Noetherian type.

1. INTRODUCTION

The Noetherian type of a topological space is an order-theoretic analog of its weight.

Definition 1.1. Given a cardinal κ , define a poset to be κ^{op} -like if no element is below κ -many elements.

In the context of families of subsets of a topological space, we will always implicitly order by inclusion. For example, a descending chain of open sets of type ω is ω^{op} -like; an ascending chain of open sets of type ω is ω_1^{op} -like but not ω^{op} -like.

Definition 1.2. Given a space X ,

- the *weight* of X , or $w(X)$, is the least $\kappa \geq \omega$ such that X has a base of size at most κ ;
- the *Noetherian type* of X , or $Nt(X)$, is the least $\kappa \geq \omega$ such that X has a base that is κ^{op} -like.

Equivalently, $Nt(X)$ is the least $\kappa \geq \omega$ such that X has a base \mathcal{B} such that $\bigcap \mathcal{A}$ has empty interior for all $\mathcal{A} \in [\mathcal{B}]^\kappa$.

Noetherian type was introduced by Peregudov [10]. Preceding this introduction are several papers by Peregudov, Šapirovskiĭ and Malykhin [5, 8, 9, 11] about the topological properties $Nt(\cdot) = \omega$ and $Nt(\cdot) \leq \omega_1$. More recently, the author has extensively investigated the Noetherian type of $\beta\mathbb{N} \setminus \mathbb{N}$ [7] and the Noetherian types of homogeneous compacta and dyadic compacta [6]. (See Engelking [1], Juhász [3], and Kunen [4] for all undefined terms.)

A surprising result from [6] is that no dyadic compactum has Noetherian type ω_1 . In other words, given an ω_1^{op} -like base of a dyadic compactum X , one can construct

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an ω^{op} -like base of X . This result does not generalize to all compacta. In that same paper, it was shown how to construct a compactum with Noetherian type κ , for any infinite cardinal κ . It is still an open problem whether any infinite cardinals other than ω_1 are excluded from the spectrum of Noetherian types of dyadic compacta, although it was shown that there are dyadic compacta with Noetherian ω and dyadic compacta with Noetherian type κ^+ , for every infinite cardinal κ with uncountable cofinality.

Question 1.3. If κ is a singular cardinal with cofinality ω , then is there a dyadic compactum with Noetherian type κ^+ ? Is there a dyadic compactum with weakly inaccessible Noetherian type?

The above two questions are typical of the “sup=max” problems of set-theoretic topology. See Juhász [3] for a systematic study of these problems.

Though the above two questions remain open problems, we can now answer the corresponding questions for compact linear orders. The spectrum of Noetherian types of linearly ordered compacta includes ω , excludes ω_1 , includes all singular cardinals, includes κ^+ for all uncountable cardinals κ , and excludes all weakly inaccessible cardinals. In the process of proving this claim, we will prove a general technical lemma which says roughly that if X is a product of not-too-many μ -compact GO-spaces for some fixed cardinal μ , then $d(X) \leq Nt(X)$ and in most cases $d(X) < Nt(X)$.

Definition 1.4.

- A space X is κ -compact if κ is a cardinal and every open cover of X has a subcover of size less than κ .
- A *GO-space*, or generalized ordered space, is a subspace of a linearly ordered topological space. Equivalently, a GO-space is a linear order with a topology that has a base consisting only of convex sets.
- The *density* $d(X)$ of a space X is the least infinite cardinal κ such that X has a dense subset of size at most κ .

It is natural to ask what happens to the spectrum of Noetherian types of compact linear orders if we gently relax the assumption of compactness. It turns out that there are Lindelöf linear orders with Noetherian type ω_1 , and, less expectedly, that it is consistent (relative to existence of an inaccessible cardinal) that there is a Lindelöf linear order with weakly inaccessible Noetherian type. However, it is not consistent for a Lindelöf GO-space to have strongly inaccessible Noetherian type.

We also consider the relationship between metrizability and Noetherian type, focusing on GO-spaces. Every metric space has Noetherian type ω . For a Lindelöf GO-space X , X is metrizable if and only if $Nt(X) = \omega$ if and only if $Nt(X) = \omega_1$ and X is separable. For a countable product X of compact linear orders, X is metrizable if and only if $Nt(X) = \omega$ if and only if $Nt(X) = \omega_1$. (Note that every Lindelöf metric space is separable, and every compact GO-space is a compact linear order.)

2. SMALL DENSITIES AND LARGE NOETHERIAN TYPES

Definition 2.1. The π -weight $\pi(X)$ of a space X is the least infinite cardinal κ such that a space has π -base of size at most κ ;

Proposition 2.2. [10] *If X is a space and $\pi(X) < \text{cf } \kappa \leq \kappa \leq w(X)$, then $Nt(X) > \kappa$.*

Proof. Suppose \mathcal{A} is a base of X and \mathcal{B} is π -base of X of size at most $\pi(X)$. We then have $|\mathcal{A}| \geq \kappa$; hence, there exist $\mathcal{U} \in [\mathcal{A}]^\kappa$ and $V \in \mathcal{B}$ such that $V \subseteq \bigcap \mathcal{U}$. Hence, there exists $W \in \mathcal{A}$ such that $W \subseteq V \subseteq \bigcap \mathcal{U}$; hence, \mathcal{A} is not κ^{op} -like. \square

Note that if X is a product of at most $d(X)$ -many GO-spaces, then $\pi(X) = d(X)$ is witnessed by the following construction. For any D (topologically) dense in X and of minimal size, collect all the finitely supported products of topologically open intervals with endpoints from the union of $\{\pm\infty\}$ and the set of all coordinates of points from D .

Trivially, $Nt(X) \leq w(X)^+$ for all spaces X . The next example shows that this upper bound is attained.

Example 2.3. [6] The double-arrow space, defined as $((0, 1] \times \{0\}) \cup ([0, 1) \times \{1\})$ ordered lexicographically, has π -weight ω and weight 2^{\aleph_0} . By Proposition 2.2, it has Noetherian type $(2^{\aleph_0})^+$.

3. LINDELÖF GO-SPACES

Theorem 3.1. *Every metric space has an ω^{op} -like base.*

Proof. Let X be a metric space. For each $n < \omega$, let \mathcal{A}_n be a locally finite open refinement of the (open) balls of radius 2^{-n} in X . Set $\mathcal{A} = \bigcup_{n < \omega} \mathcal{A}_n \setminus \{\emptyset\}$. The set \mathcal{A} is a base of X because if $p \in X$ and $n < \omega$, then there exists $U \in \mathcal{A}_{n+1}$ such that $p \in U$ and U is contained in the ball of radius 2^{-n} with center p . Let us show that \mathcal{A} is ω^{op} -like. Suppose that $m < \omega$, $U \in \mathcal{A}$, $V \in \mathcal{A}_m$, and $U \subsetneq V$. There then exist $p \in U$ and $\epsilon_0 > \epsilon_1 > 0$ such that the ϵ_0 -ball with center p is contained in U and the ϵ_1 -ball with center p intersects only finitely many elements of \mathcal{A}_n for all $n < \omega$ satisfying $2^{-n} > \epsilon_0/2$. If $2^{-m} \leq \epsilon_0/2$, then V is contained in the ϵ_0 -ball with center p , in contradiction with $U \subsetneq V$. Hence, $2^{-m} > \epsilon_0/2$; hence, there are only finitely many possibilities for m and V given U , for V intersects the ϵ_1 -ball with center p . \square

Lemma 3.2. *Let X be a Lindelöf GO-space with open cover \mathcal{A} . The cover \mathcal{A} has a countable, locally finite refinement consisting only of countable unions of open convex sets.*

Proof. Let $\{A_n : n < \omega\}$ be a countable refinement of \mathcal{A} consisting only of open convex sets. For each $n < \omega$, set $B_n = A_n \setminus \bigcup_{m < n} A_m$; set $\mathcal{B} = \{B_n : n < \omega\}$. The set \mathcal{B} is a locally finite refinement of \mathcal{A} . Let \mathcal{C} be the set of open convex subsets of X which intersect only finitely many elements of \mathcal{B} . Let \mathcal{D} be the set of $U \in \mathcal{C}$ satisfying $\overline{U} \subseteq V$ for some $V \in \mathcal{C}$. Let $\{D_n : n < \omega\}$ be a countable subcover of \mathcal{D} . For each $n < \omega$, set $E_n = D_n \setminus \bigcup_{m < n} D_m$; set $\mathcal{E} = \{E_n : n < \omega\}$. The set \mathcal{E} is a locally finite refinement of \mathcal{C} . For each $n < \omega$, set $F_n = A_n \setminus \bigcup \{E \in \mathcal{E} : B_n \cap E = \emptyset\}$, which is a countable union of convex sets; set $\mathcal{F} = \{F_n : n < \omega\}$. Since \mathcal{E} is locally finite, each F_n is open. Hence, each F_n is a countable union of open convex sets. Moreover, $B_n \subseteq F_n \subseteq A_n$ for all $n < \omega$; hence, \mathcal{F} is a refinement of \mathcal{A} .

Thus, it suffices to show that \mathcal{F} is locally finite. Since \mathcal{E} is a locally finite cover of X , it suffices to show that each element of \mathcal{E} only intersects finitely many elements of \mathcal{F} . Let $i < \omega$ and choose $V \in \mathcal{C}$ such that $E_i \subseteq V$. Suppose $j < \omega$ and $E_i \cap F_j \neq \emptyset$. We then have $E_i \cap B_j \neq \emptyset$ by definition of F_j . Hence, $V \cap B_j \neq \emptyset$; hence, there are only finitely many possibilities for B_j ; hence, there are only finitely many possibilities for F_j . \square

Definition 3.3. Let H_θ denote the set of all sets that are hereditarily of size less θ , where θ is a regular cardinal sufficiently large for the argument at hand. The relation $M \prec H_\theta$ means that $\langle M, \in \rangle$ is an elementary substructure of $\langle H_\theta, \in \rangle$.

Lemma 3.4. *Let X be a nonseparable, Lindelöf GO-space. The space X does not have an ω^{op} -like base.*

Proof. Let \mathcal{A} be a base of X . Let us show that \mathcal{A} is not ω^{op} -like. First, let us construct sequences of open sets $\langle A_{n,k} \rangle_{n,k < \omega}$ and $\langle B_{n,k} \rangle_{n,k < \omega}$. Our requirements are that $B_{n,i} \subseteq A_{n,i} \in \mathcal{A}$, that $B_{n,i}$ is a countable union of open convex sets, that $\{B_{n,k} : k < \omega\}$ is a locally finite cover of X and pairwise \subseteq -incomparable, and that $\{A_{i,k} : k < \omega\} \cap \{A_{j,k} : k < \omega\} \subseteq [X]^1$ for all $i < j < \omega$ and $n < \omega$.

Suppose $n < \omega$ and we are given $\langle A_{m,k} \rangle_{k < \omega}$ and $\langle B_{m,k} \rangle_{k < \omega}$ for all $m < n$ and they meet our requirements. Let $p \in X$. Set $V_p = \bigcap \{B_{m,k} : m < n \text{ and } k < \omega \text{ and } p \in B_{m,k}\}$. The set V_p is open. If $|V_p| = 1$, then set $U_p = V_p$. If $|V_p| > 1$, then choose $U_p \in \mathcal{A}$ such that $p \in U_p \subsetneq V_p$. Set $\mathcal{U} = \{U_p : p \in X\}$. By Lemma 3.2, there exists a countable, locally finite refinement \mathcal{B}_n of \mathcal{U} consisting only of countable unions of open convex sets. Since \mathcal{B}_n is locally finite, it has no infinite ascending chains; hence, we may assume \mathcal{B}_n is pairwise \subseteq -incomparable because we may shrink \mathcal{B}_n to its maximal elements. Let $\{B_{n,k} : k < \omega\} = \mathcal{B}_n$. For each $k < \omega$, set $A_{n,k} = U_p$ for some $p \in X$ satisfying $B_{n,k} \subseteq U_p$. Suppose $m < n$ and $i, j < \omega$ and $A_{m,i} = A_{n,j} \notin [X]^1$. Choose $p \in X$ such that $A_{n,j} = U_p$; choose $k < \omega$ such that $p \in B_{m,k}$. We then have $B_{m,i} \subseteq A_{m,i} = A_{n,j} = U_p \subsetneq V_p \subseteq B_{m,k}$, in contradiction with the pairwise \subseteq -incomparability of $\{B_{m,l} : l < \omega\}$. Thus, $\{A_{m,l} : l < \omega\} \cap \{A_{n,l} : l < \omega\} \subseteq [X]^1$ for all $m < n$. By induction, $\langle A_{n,k} \rangle_{n,k < \omega}$ and $\langle B_{n,k} \rangle_{n,k < \omega}$ meet our requirements.

Let $\{X, \leq, \mathcal{A}\} \subseteq M \prec H_\theta$ and $|M| = \omega$. Since X is nonseparable, there must be a nonempty open convex set W disjoint from M . Since X is Lindelöf, every isolated point of X is in M . Hence, W must be infinite. Choose $\{a < c < e\} \subseteq W$. Choose $U \in \mathcal{A}$ such that $U \subseteq (a, e)$. By elementarity, we may assume that $\langle \langle A_{n,k}, B_{n,k} \rangle \rangle_{n,k < \omega} \in M$ for all $n, k < \omega$. For each $n < \omega$, choose $i_n < \omega$ such that $c \in B_{n,i_n}$. To complete the proof, it suffices to show that U has infinitely many supersets in \mathcal{A} . Fix $n < \omega$. Since $c \notin M$, we cannot have $A_{n,i_n} = \{c\}$; hence, $A_{n,i_n} \neq A_{m,i_m}$ for all $m < n$. Hence, it suffices to show that $U \subseteq A_{n,i_n}$. There exists $\langle I_j \rangle_{j < \omega} \in M$ such that $B_{n,i_n} = \bigcup_{j < \omega} I_j$ and each I_j is convex and open. Hence, there exists $j < \omega$ such that $c \in I_j$. Since $U \subseteq (a, e)$ and $I_j \subseteq B_{n,i_n} \subseteq A_{n,i_n}$, it suffices to show that $(a, e) \subseteq I_j$. Seeking a contradiction, suppose $(a, e) \not\subseteq I_j$. Since $c \in I_j$, we must have $a < b < I_j$ for some $b \in X$ or $I_j < d < e$ for some $d \in X$. By symmetry, we may assume we have the former. Since X is Lindelöf, $\{p \in X : p < I_j\}$ has a countable cofinal subset Y . Since $I_j \in M$, we may assume $Y \in M$. Hence, $Y \subseteq M$; hence, there exists $y \in M$ such that $b \leq y < I_j$. Hence, M intersects (a, e) ; hence, M intersects W , which is absurd. \square

Theorem 3.5. *Let X be a Lindelöf GO-space. The following are equivalent.*

- (1) X is metrizable.
- (2) X has an ω^{op} -like base.
- (3) X is separable and has an ω_1^{op} -like base.

Proof. By Theorem 3.1, (1) implies (2). By Lemma 3.4, (2) implies (3). Hence, it suffices to show that (3) implies (1). Suppose X has a countable dense subset D and

an ω_1^{op} -like base. We then have $\pi(X) = \omega$; hence, by Proposition 2.2, $w(X) = \omega$; hence, X is metrizable. \square

See Example 6.1 for a nonseparable Lindelöf linear order that has Noetherian type ω_1 .

4. SMALL NOETHERIAN TYPES AND SMALLER DENSITIES

For compact linearly ordered topological spaces, the theorem at the end of this section strengthens Theorem 3.5. To prepare for this theorem, we first prove our main technical lemma, which we state in very general terms.

Lemma 4.1. *Suppose κ is a regular uncountable cardinal, μ is an infinite cardinal, $|\lambda^{<\mu}| < \kappa$ for all $\lambda < \kappa$, X is a product of fewer than κ -many μ -compact GO-spaces, and $Nt(X) \leq \kappa$. We then have $d(X) < \kappa$.*

Proof. Let $X = \prod_{i < \nu} X_i$ where $\nu < \kappa$ and each X_i is a μ -compact subspace of a linearly ordered topological space Y_i . Seeking a contradiction, suppose $d(X) \geq \kappa$. Let \mathcal{U} be a κ^{op} -like base of X , $\{(Y_i, \leq_{Y_i}, X_i : i < \nu), \mathcal{U}\} \in M \prec H_\theta$, $|M| < \kappa$, $M \cap \kappa \in \kappa$, and $M^{<\mu} \subseteq M$. (We can construct M as the union of an appropriate elementary chain of length ρ , where ρ is the least regular cardinal $\geq \mu$. Such an M is not too large because $\rho < \kappa$, a fact that follows from $\mu \leq |2^{<\mu}| < \kappa$ and $\text{cf}(\mu) < \mu \Rightarrow \mu^+ \leq |\mu^{\text{cf}(\mu)}| < \kappa$). Since $d(X) > |M|$, there is a finite subproduct $\prod_{i \in \sigma} X_i$ of X that has a nonempty open subset disjoint from M . We may choose this open subset to be the interior of a set of the form $B = \prod_{i \in \sigma} B_i$ where each B_i is maximal among the convex subsets of X_i disjoint from M . Set $A_i = \{p \in X_i : p < B_i\}$ and $C_i = \{p \in X_i : p > B_i\}$. Since $\{p : A_i < p < C_i\} = B_i$, which is nonempty but disjoint from M , we have $\{A_i, C_i\} \not\subseteq M$ by elementarity.

Claim. $\max\{\text{cf}(A_i), \text{ci}(C_i)\} \geq \mu$ for all $i \in \sigma$.

Proof. Seeking a contradiction, suppose $\text{cf}(A_i) < \mu$ and $\text{ci}(C_i) < \mu$. We then have $A_i \cap M, C_i \cap M \in M$. Since $A_i \cap M$ is cofinal in A_i , $A_i = \{p : \exists q \in A_i \cap M \ p \leq q\} \in M$. Likewise, $C_i \in M$, in contradiction with the fact that $\{A_i, C_i\} \not\subseteq M$. \square

Therefore, we may assume that $\text{cf}(A_i) \geq \mu$ for all $i \in \sigma$ (by symmetry). Since X is μ -compact, there exists $x_i = \sup_{Y_i}(A_i) = \min(B_i) \in X_i$ for all $i \in \sigma$.

Claim. *There exists $y_i = \sup_{Y_i}(B_i) \in (x_i, \infty)$ for all $i \in \sigma$, with the understanding that in this proof all intervals are intervals of X_i (so $y_i \in X_i$).*

Proof. If $\text{ci}(C_i) \geq \mu$, then, by μ -compactness, there exists $y_i = \inf_{Y_i}(C_i) \in X_i$. In this case, y_i is also $\max(B_i)$ because $y_i \notin C_i$. Moreover, $y_i = \max(B_i) > \min(B_i) = x_i$ because otherwise the interior of B would be empty, for $x_i = \sup_{Y_i}(A_i)$, which is not an isolated point in X_i . If $\text{ci}(C_i) < \mu$, then $C_i \in M$, just as in the previous claim's proof, so there exists $D_i \in M$ such that D_i is a cofinal subset of $\{p \in X_i : p < C_i\}$ of minimal size. In this case, D_i includes a cofinal subset of B_i , so $D_i \not\subseteq M$, so $|D_i| \geq \kappa$, so $\mu < \kappa \leq |D_i| = \text{cf}(B_i)$, so there exists $y_i = \sup_{Y_i}(B_i) \in X_i$ by μ -compactness. Also, $\text{cf}(B_i) \geq \kappa$ implies $\sup_{Y_i}(B_i) > \min(B_i) = x_i$. Thus, in any case there exists $y_i = \sup(B_i) \in (x_i, \infty)$ for all $i \in \sigma$. \square

Let $U \in \mathcal{U}$ satisfy $x_i \in \pi_i[U] \subseteq (-\infty, y_i)$ for all $i \in \sigma$. Since $\text{cf}(A_i) \geq \mu > 1$ for all $i \in \sigma$, we then have $\langle x_i : i \in \sigma \rangle \in \prod_{i \in \sigma} W_i \subseteq \pi_\sigma[U]$ where each W_i is of the form (u_i, v_i) or $(u_i, v_i]$ for some $u_i < x_i$ and $v_i \leq y_i$; we may assume $u_i \in M$. Moreover,

there exist $p_i, q_i \in X_i \cap M$ such that $u_i < p_i < q_i < x_i$. Since \mathcal{U} is a κ^{op} -like base, it includes fewer than κ -many supersets of $\bigcap_{i \in \sigma} \pi_i^{-1}[(u_i, q_i)]$ as members. Since the set of supersets of $\bigcap_{i \in \sigma} \pi_i^{-1}[(u_i, q_i)]$ in \mathcal{U} is a set in M and a set of size less than κ , it is also a subset of M . In particular, $U \in M$.

Fix an arbitrary $i \in \sigma$. If $\text{cf}(\pi_i[U]) < \mu$, then M would include a cofinal subset of $\pi_i[U]$, in contradiction with B_i missing M . Therefore, $\text{cf}(\pi_i[U]) \geq \mu$. Hence, there exists $z = \sup_{Y_i}(\pi_i[U])$. By elementarity, $z \in M$, so $B_i < z$, so $z = \min(C_i) = \sup_{Y_i}(B_i) = y$. Because of the freedom in how we chose U , it follows that every neighborhood of x_i includes a neighborhood that, like $\pi_i[U]$, has supremum y_i (in Y_i) and has cofinality at least μ . Therefore, there is an infinite increasing sequence of points between x_i and y_i that are contained in every neighborhood of x_i , in contradiction with X_i being a subspace of the ordered space Y_i . Thus, $d(X) < \kappa$. \square

Corollary 4.2. *Suppose that X is a product of at most 2^{\aleph_0} -many Lindelöf GO-spaces such that $Nt(X) \leq (2^{\aleph_0})^+$. We then have $d(X) \leq 2^{\aleph_0}$.*

Corollary 4.3. *Suppose that κ is a regular uncountable cardinal, X is a product of less than κ -many linearly ordered compacta, and $Nt(X) \leq \kappa$. We then have $d(X) < \kappa$.*

The last corollary fails for singular κ . As we shall see in Theorem 5.1, if λ is an uncountable singular cardinal, then $Nt(\lambda+1) = \lambda$, despite the fact that $d(\kappa+1) = \kappa$ for all infinite cardinals κ . Moreover, the space $(\kappa+1)^\kappa$ has Noetherian type ω and density κ for all infinite cardinals κ , so we cannot weaken the above hypothesis that X has less than κ -many factors. The equation $Nt((\kappa+1)^\kappa) = \omega$ follows from a general theorem of Malykhin.

Theorem 4.4. [5] *Let $X = \prod_{i \in I} X_i$ where each X_i has a minimal open cover of size two (e.g., X_i is T_1). If $\sup_{i \in I} w(X_i) \leq |I|$, then $Nt(X) = \omega$.*

Proof. For each $i \in I$, let $\{U_{i,0}, U_{i,1}\}$ be a minimal open cover of X_i . Since $w(X) = \sup_{i \in I} w(X_i)$, we may choose \mathcal{A} to be a base of X of size at most $|I|$ and choose an injection $f: \mathcal{A} \rightarrow I$. Let \mathcal{B} denote the set of all nonempty sets of the form $V \cap \pi_{f(V)}^{-1}[U_{f(V),j}]$ where $V \in \mathcal{A}$ and $j < 2$. Since f is injective, every infinite subset of \mathcal{B} has empty interior. Hence, \mathcal{B} is an ω^{op} -like base of X . \square

Theorem 4.5. *Let X be a product of countably many linearly ordered compacta. The following are equivalent.*

- (1) X is metrizable.
- (2) X has an ω^{op} -like base.
- (3) X has an ω_1^{op} -like base.
- (4) X is separable and has an ω_1^{op} -like base.

Proof. By Theorem 3.1, (1) implies (2), which trivially implies (3). By Corollary 4.3, (3) implies (4). Finally, (4) implies (1) because if X is separable, then $\pi(X) = \omega$, so $w(X) = \omega$ by Proposition 2.2. \square

5. THE NOETHERIAN SPECTRUM OF THE COMPACT ORDERS

Theorem 4.5 implies that no linearly ordered compactum has Noetherian type ω_1 . What is the class of Noetherian types of linearly ordered compacta? We shall prove

that an infinite cardinal κ is the Noetherian type of a linearly ordered compactum if and only if $\kappa \neq \omega_1$ and κ is not weakly inaccessible.

Theorem 5.1. *Let κ be an uncountable cardinal and give $\kappa + 1$ the order topology. If κ is regular, then $Nt(\kappa + 1) = \kappa^+$; otherwise, $Nt(\kappa + 1) = \kappa$.*

Proof. Using Corollary 4.3, the lower bounds on $Nt(\kappa + 1)$ are easy. We have $d(\kappa + 1) \geq \lambda$ for all regular $\lambda \leq \kappa$, so $Nt(\kappa + 1) > \lambda$ for all regular $\lambda \leq \kappa$. It follows that $Nt(\kappa + 1) \geq \kappa$ and $Nt(\kappa + 1) > \text{cf } \kappa$. We can also prove these lower bounds directly using the Pressing Down Lemma. Let \mathcal{A} be a base of $\kappa + 1$ and let λ be a regular cardinal $\leq \kappa$. Let us show that \mathcal{A} is not λ^{op} -like. For every limit ordinal $\alpha < \lambda$, choose $U_\alpha \in \mathcal{A}$ such that $\alpha = \max U_\alpha$; choose $\eta(\alpha) < \alpha$ such that $[\eta(\alpha), \alpha] \subseteq U_\alpha$. By the Pressing Down Lemma, η is constant on a stationary subset S of λ . Hence, $\mathcal{A} \ni \{\eta(\min S) + 1\} \subseteq U_\alpha$ for all $\alpha \in S$; hence, \mathcal{A} is not λ^{op} -like. Once again, it follows that $Nt(\kappa + 1) \geq \kappa$ and $Nt(\kappa + 1) > \text{cf } \kappa$.

Trivially, $Nt(\kappa + 1) \leq w(\kappa + 1)^+ = \kappa^+$. Hence, it suffices to show that $\kappa + 1$ has a κ^{op} -like base if κ is singular. Suppose $E \in [\kappa]^{<\kappa}$ is unbounded in κ . Let F be the set of limit points of E in $\kappa + 1$. Define \mathcal{B} by

$$\mathcal{B} = \{(\beta, \alpha] : E \ni \beta < \alpha \in F \text{ or } \sup(E \cap \alpha) \leq \beta < \alpha \in \kappa \setminus F\}.$$

The set \mathcal{B} is a κ^{op} -like base of $\kappa + 1$. □

Definition 5.2. Given a poset P with ordering \leq , let P^{op} denote the set P with ordering \geq .

Theorem 5.3. *Suppose κ is a singular cardinal. There then is a linearly ordered compactum with Noetherian type κ^+ .*

Proof. Set $\lambda = \text{cf } \kappa$ and $X = \lambda^+ + 1$. Partition the set of limit ordinals in λ^+ into λ -many stationary sets $\langle S_\alpha \rangle_{\alpha < \lambda}$. Let $\langle \kappa_\alpha \rangle_{\alpha < \lambda}$ be an increasing sequence of regular cardinals with supremum κ . For each $\alpha < \lambda$ and $\beta \in S_\alpha$, set $Y_\beta = (\kappa_\alpha + 1)^{\text{op}}$. For each $\alpha \in X \setminus \bigcup_{\beta < \lambda} S_\beta$, set $Y_\alpha = 1$. Set $Y = \bigcup_{\alpha \in X} \{\alpha\} \times Y_\alpha$ ordered lexicographically. We then have $Nt(Y) \leq w(Y)^+ \leq |Y|^+ = \kappa^+$. Hence, it suffices to show that Y has no κ^{op} -like base.

Seeking a contradiction, suppose \mathcal{A} is a κ^{op} -like base of Y . For each $\alpha < \lambda$, let \mathcal{U}_α be the set of all $U \in \mathcal{A}$ that have at least κ_α -many supersets in \mathcal{A} . For all isolated points p of Y , there exists $\alpha < \lambda$ such that $\{p\} \notin \mathcal{U}_\alpha$; whence, $p \notin \bigcup \mathcal{U}_\alpha$. Since $\langle \alpha + 1, 0 \rangle$ is isolated for all $\alpha < \lambda^+$, there exist $\beta < \lambda$ and a set E of successor ordinals in λ^+ such that $|E| = \lambda^+$ and $(E \times 1) \cap \bigcup \mathcal{U}_\beta = \emptyset$. Let C be the closure of E in λ^+ . The set C is closed unbounded; hence, there exists $\gamma \in C \cap S_{\beta+1}$. Set $q = \langle \gamma, \kappa_{\beta+1} \rangle$. We then have $q \in \overline{E \times 1}$; hence, $q \notin \bigcup \mathcal{U}_\beta$. Since q has coinitality $\kappa_{\beta+1}$, any local base \mathcal{B} at q will contain an element U such that U has κ_β -many supersets in \mathcal{B} . Hence, there exists $U \in \mathcal{U}_\beta$ such that $q \in U$; hence, $q \in \bigcup \mathcal{U}_\beta$, which yields our desired contradiction. □

Theorem 5.4. *No linearly ordered compactum has weakly inaccessible Noetherian type. More generally, for every weakly inaccessible κ , products of fewer than κ -many linearly ordered compacta do not have Noetherian type κ .*

Proof. Suppose κ is weakly inaccessible, X is a product of fewer than κ -many linearly ordered compacta, and $Nt(X) \leq \kappa$. It suffices to prove $Nt(X) < \kappa$. By Corollary 4.3, we have $d(X) < \kappa$; hence, each factor of X has π -weight less

than κ ; hence, $\pi(X) < \kappa$. If $w(X) \geq \kappa$, then $Nt(X) > \kappa$ by Proposition 2.2, in contradiction with our assumptions about X . Hence, $w(X) < \kappa$; hence, $Nt(X) \leq w(X)^+ < \kappa$. \square

6. THE LINDELÖF SPECTRUM

The spectrum of Noetherian types of Lindelöf linearly ordered topological spaces trivially includes the spectrum of Noetherian types of compact linearly ordered topological spaces. More interestingly, the inclusion is strict, as the next example shows.

Example 6.1. Theorem 4.5 fails for Lindelöf linearly ordered topological spaces. Let X be $(\omega_1 \times \mathbb{Z}) \cup (\{\omega_1\} \times \{0\})$ ordered lexicographically. The space X is Lindelöf and nonseparable and $\{\langle \alpha, n \rangle : \alpha < \omega_1 \text{ and } n \in \mathbb{Z}\} \cup \{X \setminus (\alpha \times \mathbb{Z}) : \alpha < \omega_1\}$ is an ω_1^{op} -like base of X . Moreover, X has no ω^{op} -like base because every local base at $\langle \omega_1, 0 \rangle$ includes a descending ω_1 -chain of neighborhoods. Thus, $Nt(X) = \omega_1$. Easily generalizing this example, if κ is a regular cardinal and X is $(\kappa \times \mathbb{Z}) \cup (\{\kappa\} \times \{0\})$ ordered lexicographically, then X is κ -compact and $Nt(X) = \kappa$.

A consequence of Lemma 4.1 is that Lindelöf linearly ordered topological spaces cannot have strongly inaccessible Noetherian type, just as in the compact case. More generally, we have the following theorem, which is proved just as Theorem 5.4 was proved.

Theorem 6.2. *Suppose κ is a weakly inaccessible cardinal, $|\lambda^{<\mu}| < \kappa$ for all $\lambda < \kappa$, and X is a product of fewer than κ -many μ -compact GO-spaces. We then have $Nt(X) \neq \kappa$.*

Proof. Suppose that $Nt(X) \leq \kappa$. Let us show that $Nt(X) < \kappa$. By Lemma 4.1, we have $d(X) < \kappa$; hence, each factor of X has π -weight less than κ ; hence, $\pi(X) < \kappa$. If $w(X) \geq \kappa$, then $Nt(X) > \kappa$ by Proposition 2.2, in contradiction with our assumptions about X . Hence, $w(X) < \kappa$; hence, $Nt(X) \leq w(X)^+ < \kappa$. \square

Corollary 6.3. *If κ is strongly inaccessible, then the class of Noetherian types of μ -compact GO-spaces excludes κ if and only if $\mu < \kappa$.*

Proof. “If”: Theorem 6.2. “Only if”: Example 6.1. \square

On the other hand, it is consistent (relative to the consistency of an inaccessible), that some Lindelöf linearly ordered topological space has weakly inaccessible Noetherian type. To show this, we first force $2^{\aleph_0} \geq \kappa$ where κ is weakly inaccessible (say, by adding κ -many Cohen reals). Next, we construct the desired linear order in this forcing extension using the following theorem.

Theorem 6.4. *If κ is a weakly inaccessible and $2^{\aleph_0} \geq \kappa$, then there is a Lindelöf linear order Z such that $Nt(Z) = \kappa$.*

Proof. Let B be a Bernstein subset of $X = [0, 1]$, i.e., B includes some point in P and misses some point in P , for all perfect $P \subseteq X$. Let $f: B \rightarrow \kappa$ be surjective. For each $x \in B$, set $Y_x = \omega^{\text{op}} + \omega_{f(x)} + \omega$, which is Lindelöf. For each $x \in X \setminus B$, set $Y_x = \{0\}$. Set $Z = \bigcup_{x \in X} (\{x\} \times Y_x)$ ordered lexicographically. First, let us show that Z is Lindelöf. Let \mathcal{U} be an open cover of Z . For every $x \in X \setminus B$, $\langle x, 0 \rangle$ has neighborhoods O_x and U_x such that $\mathcal{U} \ni U_x \supseteq O_x = \bigcup_{a < b < c} (\{b\} \times Y_b)$ where $a, c \in (X \cap \mathbb{Q}) \cup \{\pm\infty\}$. Therefore, there is a countable $D \subseteq X$ such that $\{O_x : x \in D\}$

covers $(X \setminus B) \times \{0\}$. Set $\mathcal{V} = \{U_x : x \in D\}$ and $C = \{x \in X : Y_x \not\subseteq \bigcup_{x \in D} O_x\}$. The set C is closed in X and a subset of the Bernstein set B , so C is countable. Therefore, $\bigcup_{x \in C} (\{x\} \times Y_x)$ is Lindelöf; hence, it is covered by a countable $\mathcal{W} \subseteq \mathcal{U}$, making $\mathcal{V} \cup \mathcal{W}$ a countable subcover of \mathcal{U} .

Finally, let us show that $Nt(Z) = \kappa$. For every $\alpha < \kappa$ and $x \in f^{-1}[\{\alpha + 1\}]$, Y_x has a point with cofinality $\omega_{\alpha+1}$, so $Nt(Z) \geq \omega_{\alpha+1}$. Therefore, it suffices to construct a κ^{op} -like base of Z . Let \mathcal{A} denote the countable set of all sets of the form $\bigcup_{a < b < c} Y_b$ where $a, c \in (X \cap \mathbb{Q}) \cup \{\pm\infty\}$, which includes a local base at $\langle x, 0 \rangle$ for every $x \in X \setminus B$. Since each Y_x for $x \in B$ has no maximum or minimum, we can combine \mathcal{A} with a copy of a base \mathcal{B}_x of Y_x for each $x \in B$ in order to produce a base \mathcal{B} of Z . We may choose each \mathcal{B}_x to have size less than κ , so \mathcal{B} must be κ^{op} -like. \square

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